Exercises for 'Topics in complex analysis'

(17/12/2025)

H 14.1 (A higher dimensional Schwarz lemma)
Let $f : B_1(0) \subset \mathbb{C}^n \to \mathbb{C}$ be holomorphic such that $f(0) = 0$. Assume that there exists a constant $M > 0$ such that $|f(z)| \leq M$ for all $z \in B_1(0)$.

a) Show that

$$|f(z)| \leq M\|z\| \qquad \forall z \in B_1(0),$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{C}^n$.

b) If $n = 1$, the equality $|f(z)| = M|z|$ for some $z \neq 0$ implies that $f(z) = Maz$ for some $a \in \partial B_1(0)$, so f is biholomorphic. Show that for $n \geq 2$ there exists a holomorphic function $f : B_1(0) \to B_1(0)$ with $f(0) = 0$ and $\|f(z)\| = \|z\|$ for some $z \in B_1(0) \setminus \{0\}$ that is not even injective.

Remark: Cartan's uniqueness theorem states that if $D \subset \mathbb{C}^n$ is a bounded domain and $f : D \to D$ has a fixed point $a \in D$ with $Df(a) = \mathrm{Id}$ then $f(z) = z$ for all $z \in D$.

H 14.2 (A stronger version of the identity theorem)
Let $D \subset \mathbb{C}^n$ be a domain and $f : D \to \mathbb{C}$ be holomorphic. Assume that there exists $a \in D$ such that for every multi-index $\alpha \in (\mathbb{N}_0)^n$ it holds that $D^\alpha f(a) = 0$. Show that $f \equiv 0$.

H 14.3 (Consequences of Hartogs's extension theorem)
Let $n \geq 2$ and $U \subset \mathbb{C}^n$ be open. Show the following statements.

a) If $f : U \setminus \{a\} \to \mathbb{C}$ is holomorphic, then f can be extended to a holomorphic function $f : U \to \mathbb{C}$.

b) If $K \subset \mathbb{C}^n$ is compact and such that $\mathbb{C}^n \setminus K$ is connected, then every holomorphic function $f : \mathbb{C}^n \setminus K \to \mathbb{C}$ can be extended to an entire function.

c) If $f : U \to \mathbb{C}$ is holomorphic, then f cannot have an isolated zero.

d) If $f : \mathbb{C}^n \to \mathbb{C}$ is entire, then $\{f = 0\}$ is either empty or unbounded.